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Restricted flows of soliton hierarchies: coupled KdV and Harry Dym case

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Abstract. Restricted flows of soliton hierarchies associated with the energy-dependent Schrödinger spectral problem are determined explicitly. A remarkable connection with separable potentials is used for proving complete integrability of the restricted flows. A previously unknown Lagrangian and Hamiltonian formulation of the Neumann system is found. Whole families of generalizations of the Neumann and Garnier systems are given.

1. Introduction

In previous papers [1, 2] we introduced a general method of constructing finite-dimensional integrable systems related to a given bi-Hamiltonian hierarchy of soliton equations. This paper presents the results of systematic application of this method to the hierarchies of coupled KdV (cKdV) equations and coupled Harry Dym (cHD) equations. The results show a surprising fact that the theory of stationary flows of the Schrödinger spectral problem is closely connected with the classical separability theory of the Hamilton–Jacobi equation.

The main limitation in the construction of integrable finite-dimensional systems given in [1, 2] is the problem of explicit solvability of the condition $K_r = B_m\Phi_{(-m)}$ (see (2.12*b*) here) which restricts flows of the soliton hierarchy to an invariant finite-dimensional manifold. We address this question here and present a general solution for K_0 and K_1 flows. We would like to stress that many of these finite-dimensional systems are physically interesting because they describe free natural Hamiltonian systems or Newton potential equations constrained to some surface. The natural Hamiltonian systems which arise here are identified with some previously known separable potentials. The constrained Newton equations are natural generalizations of the Neumann and Garnier systems.

In order to show how our construction of restricted flows works in concrete cases we discuss first some 'few component' cKdV and cHD cases and then derive general formulae for the whole families of cKdV and cHD equations. All dynamical systems derived in this paper are expected to be completely integrable. We prove this for some families but not for all as yet.

With the multiple forms of the same dynamical system, which naturally arise from our construction, there are connected finite-dimensional multi-Hamiltonian structures

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[2] which we do not discuss in this paper except the Neumann system. For the Neumann system we find here a previously unknown Lagrangian description which simultaneously contains information that the system is constrained to a sphere. From this Lagrangian we derive a new Hamiltonian formulation which on the constraints is equivalent to the Moser [3] Hamiltonian formulation of the Neumann system.

We begin in section 2 with multi-Hamiltonian formulation of the cKdV and cHD hierarchies. Then the square eigenfunction relations are introduced and the restricted flows are defined. The problem of solvability of the restriction (2.12*b*) is discussed. In section 3 we study restricted flows of the cKdV hierarchy. First for the KdV, DDW and Ito hierarchies (section 3.1) and then in the whole generality for K_0 and K_1 restricted flows in section 3.2. Section 4 about cHD restricted flows follows the same pattern as section 3. It starts with the few component cases $N=1,2$ and then gives general solutions for the K_0 and K_1 restricted flows. In the appendices we give complete characterization of the Neumann family of separable potentials and discuss new Lagrangian and Hamiltonian formulation of the Neumann system.

2. Restricted flows of the energy-dependent Schrödinger spectral problem

2.1. *Energy-dependent Schrödinger spectral problem: formulation and notation*

Let us consider the spectral problem [4]

$$L(u,\lambda)\phi=\left\{\left(\sum_{i=0}^{N}\varepsilon_i\lambda^i\right)\partial^2+\left(\sum_{i=0}^{N}u_i\lambda^i\right)\right\}\phi\equiv(\varepsilon(\lambda)\partial^2+u)\phi \tag{2.1}$$

where $\partial=\partial/\partial x$, ε_i are constants and u_i are functions of x. It is usually called the energy-dependent Schrödinger spectral problem since the potential u depends on the spectral parameter λ. Compatibility of (2.1) with the time evolution equations

$$\phi_t=(\tfrac{1}{2}P(u,\lambda)\partial+Q(u,\lambda))\phi$$

for the wavefunction ϕ gives rise to $Q=-\frac{1}{4}P_x$ and

$$u_t=\{\tfrac{1}{4}\varepsilon\partial^3+\tfrac{1}{2}(u\partial+\partial u)\}P\equiv\left(\sum_{i=0}^{N}J_i\lambda^i\right)P\equiv JP \tag{2.2}$$

where $J_i=\frac{1}{4}\varepsilon_i\partial^3+\frac{1}{2}(u_i\partial+\partial u_i)$ and the subscripts t and x mean derivatives. If one assumes $P=F+P_{m-1}\lambda+\ldots+P_0\lambda^m$ then (2.2) leads (at powers $\lambda^m,\ldots,\lambda^{m+N}$) to m recursion equations:

$$J_0P_{k-N}+J_1P_{k-N+1}+\ldots+J_NP_k=0 \qquad k=0,\ldots,m-1 \tag{2.3}$$

for the m unknowns $P_0,\ldots,P_{m-1}$ ($P_r=0$ for $r=-1,-2,\ldots$ in (2.3)). At the remaining powers $\lambda^0,\ldots,\lambda^N$ one gets $N+1$ evolution equations:

$$\begin{pmatrix}u_0\\ \vdots\\ u_N\end{pmatrix}_{t_m}=\begin{pmatrix}0&\cdots&0&J_0\\ \vdots&\cdot^{\cdot^{\cdot}}&\cdot^{\cdot^{\cdot}}&\vdots\\ 0&J_0&&J_{N-1}\\ J_0&\cdots&J_{N-1}&J_N\end{pmatrix}\begin{pmatrix}P_{m-N}\\ \vdots\\ P_{m-1}\\ F\end{pmatrix}\equiv K_m. \tag{2.4}$$

$K_m[u]$ denotes here the vector field of the t_m flow. As has been shown [4], recursion equations (2.3) can be solved starting with $P_0\in\ker(J_N)$ iff $\varepsilon_N=0$. F in (2.4) remains underdetermined by (2.3) and has to be chosen consistently with the structure of equations (2.4). Thus (2.4) splits into two cases:

(i) *Coupled* KdV *case* when $u_N = \text{constant}$ (set $u_N = -1$). The last equation (2.4) complements equations (2.3) to determine $F = P_m$. The remaining equations (2.4) then take the $N+1$ Hamiltonian form

$$\begin{pmatrix} u_0 \\ \vdots \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} 0 & \dots & J_0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ J_0 & \dots & J_{N-1} \end{pmatrix} \begin{pmatrix} P_{m-N+1} \\ \vdots \\ P_m \end{pmatrix} = \dots = \begin{pmatrix} -J_1 & \dots & -J_N \\ \vdots & \cdot^{\cdot^{\cdot}} & \\ -J_N & \dots & 0 \end{pmatrix} \begin{pmatrix} P_{m+1} \\ \vdots \\ P_{m+N} \end{pmatrix}$$

$$\equiv B_N P^{(m)} \equiv \dots \equiv B_0 P^{(m+N)} \tag{2.5}$$

where for $k = 0, \dots, N$

$$B_k = \left(\begin{array}{ccc|ccc} 0 & & J_0 & & & \\ & \cdot^{\cdot^{\cdot}} & \vdots & & 0 & \\ J_0 & \dots & J_{k-1} & & & \\ \hline & & & -J_{k+1} & \dots & -J_N \\ & 0 & & \vdots & \cdot^{\cdot^{\cdot}} & \\ & & & -J_N & & 0 \end{array}\right) \tag{2.6}$$

are matrix Hamiltonian operators, $P^{(k)} = (P_{k-N+1}, \dots, P_k)^t$ and for any n the P_n can be determined through the solution of the formal series equation

$$J\left(\sum_{k=0}^{\infty} P_k \lambda^{-k}\right) = 0 \tag{2.7}$$

which is equivalent to the recursion (2.3) for arbitrary k.

(ii) *Coupled Harry Dym case* when $u_N \neq \text{constant}$ and F is unconstrained. The choice $F = 0$ implies $u_0 = -a^2 = \text{constant}$. The remaining equations (2.4) give rise to

$$\begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}_{t_m} = \begin{pmatrix} 0 & & J_0 \\ & \cdot^{\cdot^{\cdot}} & \vdots \\ J_0 & \dots & J_{N-1} \end{pmatrix} \begin{pmatrix} P_{m-N} \\ \vdots \\ P_{m-1} \end{pmatrix} = \dots = \begin{pmatrix} -J_1 & \dots & -J_N \\ \vdots & \cdot^{\cdot^{\cdot}} & \\ -J_N & & 0 \end{pmatrix} \begin{pmatrix} P_m \\ \vdots \\ P_{m+N-1} \end{pmatrix}$$

$$\equiv B_N P^{(m-1)} \equiv \dots \equiv B_0 P^{(m+N-1)} \tag{2.8}$$

which again have $(N+1)$-Hamiltonian form with the same B_k as in (2.6) and P_n determined as solutions of (2.7).

In both cases we solve the same recursion equation (2.7)

$$0 = (J_N \lambda^N + \dots + J_1 \lambda + J_0)(P_0 + P_1 \lambda^{-1} + P_2 \lambda^{-2} + \dots)$$

which starts with $0 = J_N P_0$ and gives

$$P_0 = \sqrt{(c/u_N)} \qquad P_1 = \tfrac{1}{2} u_N^{-1}[-u_{N-1} P_0 - \tfrac{1}{2}\varepsilon_{N-1} P_{0xx} + \tfrac{1}{4}\varepsilon_{N-1}(P_{0x}^2/P_0)] \dots.$$

In the cKdV case $u_N = -1$ and the choice $c = -4$ gives $P_0 = 2$, $P_1 = u_{N-1}, \dots$. In the cHD case $u_N \neq \text{constant}$ and we start with $P_0 = 1/\sqrt{u_N}$. It has been proved in [4], for cKdV and cHD cases, that the vectors $P^{(k)}$, $k = 0, 1, \dots$ are variational derivatives of a sequence of corresponding Hamiltonian densities.

2.2. *Square eigenfunction relations* (SER)

Let ϕ, ψ, be any two wavefunctions satisfying the linear equations (2.1)

$$0 = (\varepsilon(\lambda)\partial^2 + u)\phi \qquad 0 = (\varepsilon(\lambda)\partial^2 + u)\psi \tag{2.9}$$

for the same value of the spectral parameter λ. The function $\phi\psi$ satisfies the SER

$$J(\phi\psi)=\{\tfrac{1}{4}\varepsilon(\lambda)\partial^3+\tfrac{1}{2}(u\partial+\partial u)\}(\phi\psi)=0 \qquad (2.10)$$

which can be written in N equivalent ways:

$$B_m(\lambda^{-m}\phi\psi,\ldots,\lambda^{N-m-1}\phi\psi)^t$$
$$=B_{m+1}(\lambda^{-m-1}\phi\psi,\ldots,\lambda^{N-m-2}\phi\psi)^t \qquad m=0,\ldots,N-1 \qquad (2.11)$$

using the Hamiltonian operators B_m. The mth row in (2.11) is equivalent to (2.10) while the equalities in the remaining rows are satisfied identically.

Any two wavefunctions satisfying (2.9) are connected by the Wronskian relation $\phi_x\psi-\psi_x\phi=$ constant which is an integral of motion of (2.9) independently of the form of the potential u. Thus $\phi_x\psi-\psi_x\phi$ is a generator of symmetry of (2.9). In order to get rid of this symmetry we shall be interested in future only in the admissible reduction $\phi=\psi$.

2.3. Restricted flows

Restricted flows of the energy-dependent Schrödinger linear problem (2.1) have been defined in [2] as the following set of $n+N$ equations for $n+N$ unknown functions ϕ_k, u_i:

$$\varepsilon(\lambda_k)\phi_{kxx}+\left(\sum_{i=0}^{N}\lambda_k^i u_i\right)\phi_k=0 \qquad k=1,\ldots,n \qquad (2.12a)$$

$$K_r[u]=B_m[u]\Phi_{(-m)} \qquad r, m-\text{fixed} \qquad (2.12b)$$

where $\Phi_{(m)}=((\phi\Lambda^m\phi),\ldots,(\phi\Lambda^{N-1+m}\phi))^t$

$$(\phi\Lambda^j\psi)=\sum_{i=1}^{n}\lambda_i^j\phi_i\psi_i \qquad \text{for } \Lambda=\operatorname{diag}(\lambda_1,\ldots,\lambda_n).$$

In equation (2.12a) there are only N independent fields u_i since $u_N=-1$ for cKdV and $u_0=-a^2=$ constant for cHD equations. If we know explicit solutions $u(\phi)$ to equation (2.12b) (at least for some r and m) then (2.12a) becomes an autonomous second-order system of differential equations for ϕ_k. We are interested in the final form of these equations since they often have the form of Newton equations with a potential force. These potentials are later identified with families of potentials which are separable (in the sense of the Hamilton–Jacobi equation) in generalized elliptic and spherical conical coordinates.

Equations (2.12) give rise to a large variety of restricted flows which are defined by

(i) specification of N and $\varepsilon(\lambda)$;

(ii) choice of K_r and B_m.

The resulting flows depend on n parameters $\lambda_1,\ldots,\lambda_n$ which are assumed to be non-zero and all different. The choice of K_r and B_m is limited by the requirement that (2.12b) can be explicitly solved for u in terms of ϕ. For fixed K_r different B_m lead, due to the SER (2.11), to different but equivalent forms of the restricted flows. Existence of equivalent formulations is a source of bi-Hamiltonian structure for some of the restricted flows.

The KdV hierarchy corresponds to $N=1$, $\varepsilon=\varepsilon_0=1$, $B_0=\partial$, $B_1=\frac{1}{4}\partial^3+u_0\partial+\frac{1}{2}u_{0x}$ and for $r=0,1,2$ restrictions (2.12b) are given by

$$0=K_0=B_0(\phi\phi)=B_1(\phi\Lambda^{-1}\phi) \tag{2.13a}$$

$$u_{0x}=K_1=B_0(\phi\phi)=B_1(\phi\Lambda^{-1}\phi) \tag{2.13b}$$

$$\tfrac{1}{4}u_{0xxx}+\tfrac{3}{2}u_0u_{0x}=K_2=B_0(\phi\phi)=B_1(\phi\Lambda^{-1}\phi). \tag{2.13c}$$

Equation (2.13a) with B_0 gives the well known [3] Neumann system while (2.13a) with B_1 provides a different and Lagrangian (!) formulation for the Neumann system which has not been known before. This Lagrangian leads to a Hamiltonian form of the Neumann system through the Dirac theory of constraints. Equation (2.13b) with B_0 leads to the Garnier system while the B_1 Hamiltonian structure endows the Garnier system with a bi-Hamiltonian formulation [1]. Equation (2.13c) is not explicitly solvable for u and its consequences are not studied in this paper.

In the following sections we shall find general solutions to (2.12b) for the K_0 and K_1 flows of the cKdV and cHD hierarchies and shall determine the corresponding finite-dimensional systems. They are identified with the known families of integrable potentials.

3. Coupled KdV hierarchy

Coupled KdV hierarchies start with $K_0=0$ and $K_1=u_x$. After some examples we give an explicit description of their finite-dimensional restrictions for arbitrary number of fields u_k.

3.1. Examples

3.1.1. $N=1$, $\varepsilon=1$, zeroth flow restriction: the Neumann system. Since $B_0=\partial$, $B_1=\frac{1}{4}\partial^3+u_0\partial+\frac{1}{2}u_{0x}$ the first form of the restriction (2.13a): $0=B_0(\phi\phi)=\partial(\phi\phi)$ yields $(\phi\phi)=c_0$ for $\phi=(\phi_1,\ldots,\phi_n)^t$ and the flow (2.12a) is restricted to a sphere $(\phi\phi)=c_0>0$. The elimination of the 'Lagrange multiplier' u_0 from the linear problem equation $0=\phi_{xx}+(u_0-\Lambda)\phi$ yields the Neumann system [3]

$$0=\phi_{xx}+c_0^{-1}[(\phi_x\phi_x)+(\phi\Lambda\phi)]\phi-\Lambda\phi \qquad (\phi\phi)=c_0. \tag{3.1}$$

The second form of the restriction (2.13a):

$$0=B_1(\phi\Lambda^{-1}\phi)=(\tfrac{1}{4}\partial^3+u_0\partial+\tfrac{1}{2}u_{0x})(\phi\Lambda^{-1}\phi)$$

can also be solved for u_0:

$$u_0=-\tfrac{1}{2}(\phi\Lambda^{-1}\phi)^{-1}(\phi\Lambda^{-1}\phi)_{xx}+(\phi\Lambda^{-1}\phi)^{-2}\{\tfrac{1}{4}(\phi\Lambda^{-1}\phi)_x^2+d_0\}. \tag{3.2}$$

Substitution of (3.2) in (3.1) gives the second form of the Neumann system

$$0=\phi_{xx}-\left[\frac{(\phi\Lambda^{-1}\phi_{xx})+(\phi_x\Lambda^{-1}\phi_x)}{(\phi\Lambda^{-1}\phi)^{-1}}-\frac{(\phi\Lambda^{-1}\phi_x)^2+d_0}{(\phi\Lambda^{-1}\phi)^{-2}}+\Lambda\right]\phi \tag{3.3}$$

which is a Lagrangian system with the Lagrangian

$$\mathscr{L}=\tfrac{1}{2}(\phi_x\Lambda^{-1}\phi_x)-\frac{1}{2}\frac{(\phi\Lambda^{-1}\phi_x)^2-d_0}{(\phi\Lambda^{-1}\phi)^{-1}}+\tfrac{1}{2}(\phi\phi). \tag{3.4}$$

The connection between the Neumann system and the KdV equation was first noted in [5]. A more thorough discussion of the Neumann system is shifted to appendix 2.

3.1.2. First flow restriction: the Garnier system. The restriction $u_{0x} = B_0(\phi\phi) = \partial(\phi\phi)$ yields $u_0 = (\phi\phi) + c_0$ and equation (2.12*a*) gives

$$0 = \phi_{xx} + [(\phi\phi) + c_0 - \Lambda]\phi$$

which is the Garnier system. The second restriction $u_{0x} = B_1(\phi^{-1}\phi) = (\frac{1}{4}\partial^3 + u_0\partial + \frac{1}{2}u_{0x})(\phi^{-1}\phi)$ provides a second Lagrangian formulation for the Garnier system and leads to its bi-Hamiltonian formulation [1]. The connection between the Garnier system and the KdV equation was first noted in [6].

3.1.3. $N=2$, $\varepsilon = \varepsilon_0 + \lambda\varepsilon_1$, zeroth flow restriction: DWW and Ito generalizations of the Neumann system. For $N = 2$ equations (2.12*a*) read

$$[(\varepsilon_0 + \Lambda\varepsilon_1)\partial^2 + u_0 + u_1\Lambda - \Lambda^2]\phi = 0. \tag{3.5}$$

Equations of the $N = 2$ hierarchy have three compatible Hamiltonian structures:

$$B_0 = \begin{pmatrix} -J_1 & -J_2 \\ -J_2 & 0 \end{pmatrix} \qquad B_1 = \begin{pmatrix} J_0 & 0 \\ 0 & -J_2 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 0 & J_0 \\ J_0 & J_1 \end{pmatrix}$$

where $J_0 = \frac{1}{4}\varepsilon_0\partial^3 + u_0\partial + \frac{1}{2}u_{0x}$, $J_1 = \frac{1}{4}\varepsilon_1\partial^3 + u_1\partial + \frac{1}{2}u_{1x}$, $J_2 = -\partial$ and the restrictions (2.12*b*) on the K_0 flow read

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = K_0 = B_0\begin{pmatrix} (\phi\phi) \\ (\phi\Lambda\phi) \end{pmatrix} = B_1\begin{pmatrix} (\phi\Lambda^{-1}\phi) \\ (\phi\phi) \end{pmatrix} = B_2\begin{pmatrix} (\phi\Lambda^{-2}\phi) \\ (\phi\Lambda^{-1}\phi) \end{pmatrix}.$$

The first restriction $K_0 = B_0\Phi_{(0)}$ gives

$$0 = (\phi\phi)_x \qquad 0 = \tfrac{1}{4}\varepsilon_1(\phi\phi)_{xxx} + u_1(\phi\phi)_x + \tfrac{1}{2}u_{1x}(\phi\phi) - (\phi\Lambda\phi)_x$$

which have solutions $u_1 = (2/c_0)(\phi\Lambda\phi) + c_1$, $(\phi\phi) = c_0 > 0$. After substitution of u_1 the restricted flow (3.5) takes the form

$$0 = (\varepsilon_0 + \Lambda\varepsilon_1)\phi_{xx} + \left\{u_0 + \left[\frac{2}{c_0}(\phi\Lambda\phi) + c_1\right]\Lambda - \Lambda^2\right\}\phi \qquad (\phi\phi) = c_0. \tag{3.6}$$

If $\varepsilon = 1$ (3.6) reads

$$\phi_{xx} + u_0\phi + \frac{\partial V_0(\phi)}{\partial\phi} = 0 \tag{3.7}$$

with

$$V_0 = \frac{1}{2c_0}(\phi\Lambda\phi)^2 + \tfrac{1}{2}c_1(\phi\Lambda\phi) - \tfrac{1}{2}(\phi\Lambda^2\phi) \qquad u_0 = \left[\frac{1}{c_0}(\phi_x\phi_x) - \left(\phi\frac{\partial V_0(\phi)}{\partial\phi}\right)\right].$$

Equation (3.7) describes the motion of a unit mass in the quartic potential $V_0(\phi)$ which is constrained to the sphere $(\phi\phi) = c_0$. We shall call it the DWW-Neumann system since the case $\varepsilon = \varepsilon_0 = 1$ selects the dispersive water waves (DWW) equations in the cKdV family of hierarchies [4].

For $\varepsilon = \lambda$ equation (3.6) gives

$$\Lambda\phi_{xx} + u_0\phi + \frac{\partial V_0(\phi)}{\partial\phi} = 0 \tag{3.8}$$

with

$$u_0 = \frac{1}{(\phi\Lambda^{-1}\phi)}\left[(\phi_x\phi_x) - \left(\phi\Lambda^{-1}\frac{\partial V_0(\phi)}{\partial\phi}\right)\right]$$

and V_0 as in (3.7). It does not have as direct a physical interpretation as the DWW-Neumann system.

However, using another form of the restriction, i.e. $K_0 = B_1\Phi_{(-1)}$, we end (for $\varepsilon = \lambda$) with $u_0 = d_1(\phi\Lambda^{-1}\phi)^{-2}$, $(\phi\phi) = d_0$. It leads to the finite-dimensional system

$$\phi_{xx} + u_1\phi + \frac{\partial V_1(\phi)}{\partial\phi} = 0 \tag{3.9}$$

with

$$V_1 = -\tfrac{1}{2}(\phi\Lambda\phi) - d_1(\phi\Lambda^{-1}\phi)^{-1} \qquad u_1 = \frac{1}{d_0}\left[(\phi_x\phi_x) - \left(\phi\frac{\partial V_1(\phi)}{\partial\phi}\right)\right].$$

Equations (3.9) describe the motion of a unit mass in the rational potential V_1 which is constrained to the sphere $(\phi\phi) = d_0$. We shall call it the Ito–Neumann system since $\varepsilon = \lambda$ corresponds to the Ito equation in the cKdV family [4].

From the very construction it follows that (3.8) and (3.9) are equivalent. Explicitly this equivalence is given by the proper alignment of constants c_0, c_1, d_0, d_1:

$$d_1 = (\phi\Lambda^{-1}\phi)\left[(\phi_x\phi_x) - \left(\phi\Lambda^{-1}\frac{\partial V_0(\phi)}{\partial\phi}\right)\right]$$

$$\frac{2}{c_0}(\phi\Lambda\phi) + c_1 = \frac{1}{d_0}\left[(\phi_x\phi_x) - \left(\phi\frac{\partial V_1(\phi)}{\partial\phi}\right)\right]$$

following from the form of u_0, u_1. The remaining restrictions lead to yet another form of (3.5), being instrumental for their multi-Lagrangian and multi-Hamiltonian formulation.

3.2. *General solution for cKdV hierarchy*

To find the general solution of equation (2.12*b*) we substitute it with equivalent formal series equations. This approach was introduced in [2], where the case of K_1-flow was studied in full detail. Using the K_1-flow equation as an example we briefly describe the method.

The formal series equations

$$u_x^- = -J^-\Phi^- \qquad u_x^+ = J^+\Phi^+ \tag{3.10a, b}$$

where

$$u^- = z^N u_N^- + z^{N-1}u_{N-1}^- + z^{N-2}u_{N-2}^- + \ldots$$

$$\Phi^- = \sum_{k=1}^{n} \lambda_k^-\phi_k^2 \qquad \lambda_k^- = z^{-1} + z^{-2}\lambda_k + z^{-3}\lambda_k^2 + \ldots \tag{3.11a}$$

$$J^- = \tfrac{1}{4}\varepsilon(z)\partial^3 + \tfrac{1}{2}u^-\partial + \tfrac{1}{2}\partial u^-$$

and

$$u^+ = u_0^+ + zu_1^+ + z^2u_2^+ + \ldots$$

$$\Phi^+ = \sum_{k=1}^{n} \lambda_k^+\phi_k^2 \qquad \lambda_k^+ = \lambda_k^{-1} + z\lambda_k^{-2} + z^2\lambda_k^{-3} + \ldots \tag{3.11b}$$

$$J^+ = \tfrac{1}{4}\varepsilon(z)\partial^3 + \tfrac{1}{2}u^+\partial + \tfrac{1}{2}\partial u^+$$

are equivalent to

$$u_x = B_0\Phi_{(0)} \qquad u_x = B_N\Phi_{(-N)} \tag{3.12a, b}$$

in the sense that they imply the same conditions on $u_0, \ldots, u_{N-1}$ as (3.12a, b). The advantage of using (3.10) is that they can be solved explicitly:

$$u^- = (1+\tfrac{1}{2}\Phi^-)^{-2}[c^- + \tfrac{1}{16}\varepsilon(z)(\Phi_x^-)^2 - \tfrac{1}{4}\varepsilon(z)(1+\tfrac{1}{2}\Phi^-)\Phi_{xx}^-] \tag{3.13a}$$

$$u^+ = (1-\tfrac{1}{2}\Phi^+)^{-2}[c^+ + \tfrac{1}{16}\varepsilon(z)(\Phi_x^+)^2 + \tfrac{1}{4}\varepsilon(z)(1-\tfrac{1}{2}\Phi^+)\Phi_{xx}^+] \tag{3.13b}$$

where

$$c^- = z^N c_N^- + z^{N-1}c_{N-1}^- + z^{N-2}c_{N-2}^- + \ldots \qquad c^+ = c_0^+ + zc_1^+ + z^2c_2^+ + \ldots$$

and $u_N^-, \ldots, u_0^-$ or $u_0^+, \ldots, u_{N-1}^+$ are given as the first $N+1$ (or N) terms in the series (3.13a, b) respectively. The cKdV reduction $u_N = -1$ is compatible with (3.13a) when $c_N^- = -1$.

The general solution of equation (2.12b) with arbitrary m can be built from (3.13) since B_m has the block structure and the equation

$$u_x = B_m\Phi_{(-m)} \tag{3.14}$$

consists of two separate subsystems for $u_{N-1}, \ldots, u_m$ and $u_0, \ldots, u_{m-1}$ which are of the form (3.12a) and (3.12b) respectively. Thus the solution $(u_0, \ldots, u_{N-1})$ of (3.14) can be expressed in terms of (3.13a, b) as

$$(u_0, \ldots, u_{N-1}) = (u_0^+, \ldots, u_{m-1}^+, u_m^-, \ldots, u_{N-1}^-).$$

Now we can write the general form of the system (2.12) corresponding to a fixed Hamiltonian structure B_m in (2.12b). First we divide (2.12a) by λ^m to get

$$\lambda_k^{-m}\varepsilon(\lambda_k)\phi_{kxx} + (u_0\lambda_k^{-m} + \ldots + u_N\lambda_k^{N-m})\phi_k = 0 \qquad k = 1, \ldots, n. \tag{3.15}$$

Then using (3.11) we obtain

$$u_0\lambda_k^{-m} + \ldots + u_N\lambda_k^{N-m} = \operatorname{Res} z^{-m}(\lambda_k^+u^+ + \lambda_k^-u^-)$$

where Res means residuum (the coefficient at z^{-1} in the formal power series) and u^+, u^- are given by the formulae (3.13). An important observation is that

$$\begin{aligned}\phi_k \operatorname{Res} z^{-m}[\lambda_k^+u^+ + \lambda_k^-u^-] \\ = \frac{\delta}{\delta\phi_k} \operatorname{Res} z^{-m}\{\tfrac{1}{16}\varepsilon(z)[(1+\tfrac{1}{2}\Phi^-)^{-1}(\Phi_x^-)^2 - (1-\tfrac{1}{2}\Phi^+)^{-1}(\Phi_x^+)^2] \\ + [c^+(1-\tfrac{1}{2}\Phi^+)^{-1} - c^-(1+\tfrac{1}{2}\Phi^-)^{-1}]\}.\end{aligned}$$

Thus (3.15) is the Euler–Lagrange equation $\delta\mathscr{L}_m = 0$ for

$$\mathscr{L}_m = T_m - V_m$$

with

$$\begin{aligned}T_m &= \tfrac{1}{2}(\phi_x\Lambda^{-m}\varepsilon(\Lambda)\phi_x) - \operatorname{Res} \tfrac{1}{16}\varepsilon(z)z^{-m}\{(1+\tfrac{1}{2}\Phi^-)^{-1}(\Phi_x^-)^2 - (1-\tfrac{1}{2}\Phi^+)^{-1}(\Phi_x^+)^2\} \\ V_m &= \operatorname{Res} z^{-m}\{c^+(1-\tfrac{1}{2}\Phi^+)^{-1} - c^-(1+\tfrac{1}{2}\Phi^-)^{-1}\} + \text{constant}\end{aligned} \tag{3.16}$$

where $\varepsilon(\Lambda)=\mathrm{diag}(\varepsilon(\lambda_1),\ldots,\varepsilon(\lambda_n))$. All potentials appearing in (3.16) are linear combinations of two basic families of potentials which we get by expanding $(1+\frac{1}{2}\Phi^-)^{-1}$ and $(1-\frac{1}{2}\Phi^+)^{-1}$ in powers of z. From $(1+\frac{1}{2}\Phi^-)^{-1}$ we get [2]

$$-\tfrac{1}{2}(\phi\phi) \qquad -\tfrac{1}{2}(\phi\Lambda\phi)+\tfrac{1}{4}(\phi\phi)^2 \qquad -\tfrac{1}{2}(\phi\Lambda^2\phi)+\tfrac{1}{2}(\phi\phi)(\phi\Lambda\phi)-\tfrac{1}{8}(\phi\phi)^3\ldots$$

and from $(1-\frac{1}{2}\Phi^+)^{-1}$ we get

$$M^{-1} \qquad \tfrac{1}{2}M^{-2}(\phi\Lambda^{-2}\phi) \qquad \tfrac{1}{2}M^{-2}(\phi\Lambda^{-3}\phi)+\tfrac{1}{4}M^{-3}(\phi\Lambda^{-2}\phi)^2\ldots$$

where $M=1-\frac{1}{2}(\phi\Lambda^{-1}\phi)$. They are the Jacobi family of permutationally symmetric potentials (found in [7]) which are separable in the generalized elliptic coordinates.

For our method to work in the K_0-flow case some adjustments are necessary. Equations (2.12b) now read

$$B_m\Phi_{(-m)}=0 \tag{3.17}$$

and for $m=0,\ldots,N-1$ carry the following information:

$$u_k=u_k^+ \qquad k=0,\ldots,m-1 \tag{3.18a}$$

$$u_k=u_k^- \qquad k=m+1,\ldots,N-1 \tag{3.18b}$$

$$(\phi\phi)=f_0=\text{constant}$$

where u^+ and u^- are solutions of the equations

$$J^+\Phi^+=0 \qquad J^-\Phi^-=0.$$

Substituting Φ^- with F^- given by

$$F^-=z^{-1}f_0+\sum_{k=1}^{n} l_k^-\phi_k^2 \qquad l_k^-=z^{-2}\lambda_k+z^{-3}\lambda_k^2+\ldots$$

we automatically take care of the constraint $(\phi\phi)=f_0$ and are able to write a general solution of (3.17) in the form of (3.18) with

$$u^+=(\Phi^+)^{-2}\{c^++\tfrac{1}{4}\varepsilon(z)[(\Phi_x^+)^2-2\Phi^+\Phi_{xx}^+]\} \tag{3.19a}$$

$$u^-=(F^-)^{-2}\{c^-+\tfrac{1}{4}\varepsilon(z)[(F_x^-)^2-2F^-F_{xx}^-]\}. \tag{3.19b}$$

Remark. The choice $c^-=-f_0^2z^{N-2}+c_{N-3}^-z^{N-3}+c_{N-4}^-z^{N-4}+\ldots$ in (3.19b) is compatible with the reduction $u_N=-1$.

Since

$$u_0\lambda_k^{-m}+\ldots+u_N\lambda_k^{N-m}=u_m+\mathrm{Res}\, z^{-m}(\lambda_k^+u^++l_k^-u^-)$$

and

$$\begin{aligned}\phi_k\,\mathrm{Res}\, z^{-m}(\lambda_k^+u^++l_k^-u^-)\\ &=\frac{\delta}{\delta\phi_k}\mathrm{Res}\, z^{-m}\{\tfrac{1}{8}\varepsilon(z)[(\Phi_x^+)^2(\Phi^+)^{-1}+(F_x^-)^2(F^-)^{-1}]\\ &\quad-[\tfrac{1}{2}c^+(\Phi^+)^{-1}+\tfrac{1}{2}c^-(F^-)^{-1}]\}\end{aligned}$$

equation (2.12a)

$$\begin{aligned}&\lambda_k^{-m}\varepsilon(\lambda_k)\phi_{kxx}+(u_0\lambda_k^{-m}+\ldots+u_N\lambda_k^{N-m})\phi_k=0 \qquad k=1,\ldots,n\\ &(\phi\phi)=f_0\end{aligned} \tag{3.20}$$

takes the Lagrangian form

$$\delta\{T_m - V_m - \tfrac{1}{2}u_m[(\phi\phi) - f_0]\} = 0$$

where

$$\begin{aligned} T_m &= \tfrac{1}{2}(\phi_x \Lambda^{-m}\varepsilon(\Lambda)\phi_x) - \tfrac{1}{8}\,\mathrm{Res}\, z^{-m}\varepsilon(z)[(\Phi_x^+)^2(\Phi^+)^{-1} + (F_x^-)^2(F^-)^{-1}] \\ V_m &= -\tfrac{1}{2}\,\mathrm{Res}\, z^{-m}[c^+(\Phi^+)^{-1} + c^-(F^-)^{-1}] \end{aligned} \tag{3.21}$$

and u_m is a Lagrange multiplier.

Thus we proved that for $r=0$ the system (2.12) can be written as a Lagrangian system with constraints in N equivalent ways corresponding to $m=0,\ldots,N-1$. This equivalence is realized by transformations involving constants of integration c^+, c^- (see (3.1) and [2] for the description of the case $r=1$).

If $\varepsilon(\lambda)=\lambda^s$ then equations (3.20) with $m=s$ are particularly simple:

$$\phi_{kxx} + u_s\phi_k + \frac{\partial V_s}{\partial \phi_k} = 0 \qquad k=1,\ldots,n$$

$$(\phi\phi) = f_0$$

and the Lagrange multiplier is given by

$$\begin{aligned} u_s &= f_0^{-1}\left[(\phi_x\phi_x) - \left(\phi\frac{\partial V_s}{\partial\phi}\right)\right] \\ &= f_0^{-1}(\phi_x\phi_x) + f_0^{-1}\,\mathrm{Res}\, z^{-m}\{c^-[(F^-)^{-1} - zf_0^{-1}] - c^+(\Phi^+)^{-1}\}. \end{aligned}$$

The last equality follows from the homogeneity of $[(F^-)^{-1} - zf_0^{-1}]$ and $(\Phi^+)^{-1}$ of orders 2 and -2 respectively.

The remaining case $m=N$ is slightly different. The equation

$$B_N\Phi_{(-N)} = 0$$

gives

$$u_k = u_k^+ \qquad k=0,\ldots,N-1$$

u^+ being the solution (3.19a) of $J^+\Phi^+=0$. Since

$$u_0\lambda_k^{-N} + \ldots + u_{N-1}\lambda_k + u_N = \mathrm{Res}\, z^{-N}(u^+\lambda_k^+) - 1$$

(2.12a) takes the Lagrangian form $\delta\mathscr{L}_N = 0$ with $\mathscr{L}_N = T_N - V_N$ where

$$\begin{aligned} T_N &= \tfrac{1}{2}(\phi_x\Lambda^{-N}\varepsilon(\Lambda)\phi_x) - \tfrac{1}{8}\,\mathrm{Res}\, z^{-N}\varepsilon(z)(\Phi_x^+)^2(\Phi^+)^{-1} \\ V_N &= -\tfrac{1}{2}(\phi\phi) - \tfrac{1}{2}\,\mathrm{Res}\, z^{-N}c^+(\Phi^+)^{-1}. \end{aligned} \tag{3.22}$$

There are no constraints here but the system $\delta\mathscr{L}_N=0$ is degenerate since

$$\left(\phi\frac{\partial\mathscr{L}_N}{\partial\phi_x}\right) = \left(\phi\frac{\partial T_N}{\partial\phi_x}\right) = (\phi\Lambda^{-N}\varepsilon(\Lambda)\phi_x) - \tfrac{1}{2}\,\mathrm{Res}\, z^{-N}\varepsilon(z)\Phi_x^+ = 0. \tag{3.23}$$

In appendix 2 we study this formulation of the Neumann system and compare it with Moser's description [3].

All the potentials appearing in (3.21) are linear combinations of two basic families of potentials which we get by expanding $(\Phi^+)^{-1}$ and $(F^-)^{-1}$ in powers of z. They coincide with the Neumann family of integrable potentials [7] (see appendix 1) and, therefore, describe completely integrable systems.

4. Coupled Harry Dym hierarchy

The coupled HD hierarchy corresponds to $u_0=-a^2$ reduction of (2.1). We shall present here particular solutions for $N=1,2$ and then give general formulae.

4.1. Examples

4.1.1. $N=1$, $\varepsilon=1$, zeroth flow restriction: $K_0=B_0(\phi\Lambda\phi)=B_1(\phi\phi)=0$ where $B_0=-J_1=-u_1\partial-\frac{1}{2}u_{1x}$, $B_1=\frac{1}{4}\partial^3-a^2\partial$. The equation $0=B_0(\phi\Lambda\phi)$ gives $u_1=w_1(\phi\Lambda\phi)^{-2}$, $w_1=$ constant and

$$0=\phi_{xx}+w_1(\phi\Lambda\phi)^{-2}\Lambda\phi-a^2\phi=\phi_{xx}+\frac{\partial}{\partial\phi}[-\tfrac{1}{2}w_1(\phi\Lambda\phi)^{-1}-\tfrac{1}{2}a^2(\phi\phi)]. \tag{4.1a}$$

Note that here in the restriction (2.12*b*) B_0 goes with the square eigenfunctions $\Phi_{(1)}=(\phi\Lambda\phi)$ and, consistently, B_1 goes with $\Phi_{(0)}$ in order to have a natural potential description of (4.1*a*). The other condition $0=B_1(\phi\phi)=(\frac{1}{4}\partial^3-a^2\partial)(\phi\phi)$ gives

$$\tfrac{1}{2}(\phi\phi_{xx}+\phi_x\phi_x)-a^2(\phi\phi)=w_0=\text{constant}$$

and after elimination of u_1 the restricted flow (2.12*a*) reads

$$0=\phi_{xx}+(\phi\Lambda\phi)^{-1}[(\phi_x\phi_x)-a^2(\phi\phi)-2w_0]\Lambda\phi-a^2\phi$$
$$=\phi_{xx}+2(\phi\Lambda\phi)^{-1}[E(\phi,\phi_x,w_1)-w_0]\Lambda\phi+w_1(\phi\Lambda\phi)^{-2}\Lambda\phi-a^2\phi \tag{4.1b}$$

where $E(\phi,\phi_x,w_1)=\frac{1}{2}[(\phi_x\phi_x)-w_1(\phi\Lambda\phi)^{-1}-a^2(\phi\phi)]$ is an integral of motion for (4.1*a*). Clearly equations (4.1*b*) are equivalent to (4.1*a*) if $E(\phi,\phi_x,w_1)=w_0$.

4.1.2 $N=2$, $\varepsilon=1$. Equations of this hierarchy have three compatible Hamiltonian structures:

$$B_0=\begin{pmatrix}-J_1 & -J_2\\ -J_2 & 0\end{pmatrix}\qquad B_1=\begin{pmatrix}J_0 & 0\\ 0 & -J_2\end{pmatrix}\qquad B_2=\begin{pmatrix}0 & J_0\\ J_0 & J_1\end{pmatrix}$$

where $J_0=\frac{1}{4}\partial^3-a^2\partial$, $J_1=u_1\partial+\frac{1}{2}u_{1x}$, $J_2=u_2\partial+\frac{1}{2}u_{2x}$.

The zeroth flow restriction reads

$$\begin{pmatrix}0\\0\end{pmatrix}=K_0=B_0\Phi_{(1)}=B_1\Phi_{(0)}=B_2\Phi_{(-1)}$$

where $\Phi_{(k)}=((\phi\Lambda^k\phi),(\phi\Lambda^{k+1}\phi))^t$. The condition $K_0=B_0\Phi_{(1)}$ yields

$$0=\phi_{xx}+\frac{\partial}{\partial\phi}[-w_1(\phi\Lambda\phi)^{-1}+\tfrac{1}{2}w_2(\phi\Lambda\phi)^{-2}(\phi\Lambda^2\phi)-\tfrac{1}{2}a^2(\phi\phi)] \tag{4.2a}$$

w_1, w_2 are constants, while $K_0=B_2\Phi_{(-1)}$ yields

$$\left.\begin{aligned}&0=\phi_{xx}+u_1\Lambda\phi+z_2(\phi\Lambda\phi)^{-2}-a^2\phi \qquad z_2-\text{constant}\\ &\tfrac{1}{2}(\phi\phi_{xx}+\phi_x\phi_x)-a^2(\phi\phi)=z_1=\text{constant}\end{aligned}\right\} \tag{4.2b}$$

which is equivalent to (4.2*a*).

4.2. General solution for cHD hierarchy

Hamiltonian structures of cHD hierarchies differ from those of cKdV hierarchies only through their J_0 and J_N entries which correspond to different reductions $u_0=-a^2$ and $u_N=-1$ of the linear problem (2.1). Thus the solution method for the HD version of (2.12) with $r=0$ is analogous to the cKdV case and we just introduce the necessary notation and give the final formulae.

Let

$$\Psi^-=\sum_{k=1}^{n}\zeta_k^-\phi_k^2 \qquad \zeta_k^-=z^{-1}\lambda_k+z^{-2}\lambda_k^2+\dots \tag{4.3a}$$

$$G^+=g+\sum_{k=1}^{n}p_k^+\phi_k^2 \qquad p_k^+=z\lambda_k^{-1}+z^2\lambda_k^{-2}+\dots \tag{4.3b}$$

where g is the general solution of

$$J_0g=\tfrac{1}{4}\varepsilon_0g_{xxx}-a^2g_x=0. \tag{4.3c}$$

Since $K_0=0$ (2.12b) reads

$$B_m\Phi_{(1-m)}=0$$

(note the change of convention) and for $m=1,\dots,N$ is explicitly solvable

$$u_k=u_k^+ \qquad k=1,\dots,m-1$$

$$u_k=u_k^- \qquad k=m+1,\dots,N$$

$$J_0(\phi\phi)=0 \qquad \text{(which implies } (\phi\phi)=g)$$

where

$$u^-=(\Psi^-)^{-2}\{c^-+\tfrac{1}{4}\varepsilon(z)[(\Psi_x^-)^2-2\Psi^-\Psi_{xx}^-]\} \tag{4.4a}$$

$$u^+=(G^+)^{-2}\{c^++\tfrac{1}{4}\varepsilon(z)[(G_x^+)^2-2G^+G_{xx}^+]\}. \tag{4.4b}$$

Remark. The choice

$$c_0^+=-a^2g^2-\tfrac{1}{4}\varepsilon_0(g_x^2-2gg_{xx}) \tag{4.5}$$

ensures compatibility of (4.4b) with the HD reduction $u_0=-a^2$. The right-hand side of (4.5) is a first integral of (4.3c).

The restricted flow (2.12) takes the Lagrangian form $\delta\mathscr{L}_m=0$, $m=1,\dots,N$ where

$$\begin{aligned}\mathscr{L}_m&=T_m-V_m+\tfrac{1}{2}u_m[(\phi\phi)-g]\\ T_m&=\tfrac{1}{2}(\phi_x\Lambda^{-m}\varepsilon(\Lambda)\phi_x)-\tfrac{1}{8}\operatorname{Res} z^{-m-1}\varepsilon(z)[(G^+)^{-1}(G_x^+)^2+(\Psi^-)^{-1}(\Psi_x^-)^2]\\ V_m&=-\tfrac{1}{2}\operatorname{Res} z^{-m-1}[c^+(G^+)^{-1}+c^-(\Psi^-)^{-1}]\end{aligned} \tag{4.6a}$$

and u_m is a Lagrange multiplier.

The case $m=0$ is analogous to $m=N$ in the cKdV reduction. It is described by

$$u_k=u_k^- \qquad k=1,\dots,N$$

and

$$\begin{aligned}\mathscr{L}_0&=T_0-V_0\\ T_0&=\tfrac{1}{2}(\phi_x\varepsilon(\Lambda)\phi_x)-\tfrac{1}{8}\operatorname{Res} z^{-1}\varepsilon(z)(\Psi^-)^{-1}(\Psi_x^-)^2\\ V_0&=-\tfrac{1}{2}a^2(\phi\phi)-\tfrac{1}{2}\operatorname{Res} z^{-1}c^-(\Psi^-)^{-1}.\end{aligned} \tag{4.6b}$$

There are no constraints here but if $\varepsilon_0=0$ the Lagrangian (4.6*b*) is degenerate:

$$\left(\phi\frac{\partial\mathscr{L}_0}{\partial\phi_x}\right)=\left(\phi\frac{\partial T_0}{\partial\phi_x}\right)=(\phi\varepsilon(\Lambda)\phi_x)-\tfrac{1}{2}\operatorname{Res} z^{-1}\varepsilon(z)\Psi_x^-=\varepsilon_0(\phi\phi_x).$$

The case $\varepsilon_0=0$ is distinguished since it corresponds to the degnerate form of (4.3*c*) with $g=g_0=$ constant while $\varepsilon_0=1$ leads to

$$g=g_0+w_1\,\mathrm{e}^{2ax}+w_2\,\mathrm{e}^{-2ax}\qquad a\neq 0$$

$$g=g_0+w_1x+w_2x^2\qquad a=0$$

where w_1, w_2 are additional arbitrary constants.

The basic potentials obtained from $(G^+)^{-1}$ and $(\Psi^-)^{-1}$ coincide again with the Neumann family (at least for $\varepsilon_0=0$ when $g=g_0$). It is not a surprise since

$$\Psi^-(z,\Lambda)=z^{-1}\Phi^+(z^{-1},\Lambda^{-1})$$

$$G^+(z,\Lambda)=z^{-1}F^-(z^{-1},\Lambda^{-1})\qquad \text{for } g=g_0=f_0.$$

The case of K_1-flow of the cHD hierarchy is the most complicated one. K_1 can be written as

$$u_{rt}=J_{r-1}\frac{1}{\sqrt{u_N}}\qquad r=1,\ldots,N$$

or in terms of the formal power series

$$u_t^\pm=zJ^\pm\frac{1}{\sqrt{u_N}}.$$

Thus the equation

$$K_1=B_N\Phi_{(1-N)}$$

leads to

$$u_N=[(\phi\phi)-g]^{-2}$$

$$u_k=u_k^+\qquad k=0,\ldots,N-1$$

where u^+ is given by (4.4*b*) and g satisfies $J_0g=0$. Analogously

$$K_1=B_m\Phi_{(1-m)}\qquad m=1,\ldots,N-1$$

is solved by

$$\begin{aligned}u_k&=u_k^+\qquad k=0,\ldots,m-1\\ u_k&=u_k^-\qquad k=m,\ldots,N\end{aligned}\tag{4.7}$$

where u^+,u^- are given by (4.4*a, b*) with Ψ^- substituted by

$$G^-=-g+\sum_{k=1}^{n}p_k^-\phi_k^2\qquad p_k^-=1+z^{-1}\lambda_k+z^{-2}\lambda_k^2+\ldots.$$

Remark. The solution (4.7) automatically leads to $u_0=-a^2$ and $u_N=[(\phi\phi)-g]^{-2}$ (provided c_0^+ is chosen as in (4.5)).

The restricted flow (2.12) takes the following multi-Lagrangian form:

$$\delta(T_m-V_m)=0\qquad m=1,\ldots,N$$

$$T_m=\tfrac{1}{2}(\phi_x\Lambda^{-m}\varepsilon(\Lambda)\phi_x)-\tfrac{1}{8}\operatorname{Res} z^{-m-1}\varepsilon(z)[(G^+)^{-1}(G_x^+)^2+(G^-)^{-1}(G_x^-)^2]$$

$$V_m=-\tfrac{1}{2}\operatorname{Res} z^{-m-1}[c^+(G^+)^{-1}+c^-(G^-)^{-1}].$$

Remark. The equation $K_1 = B_0\Phi_{(1)}$ cannot be solved explicitly and the K_1 restricted flow of the cHD hierarchy possesses N equivalent Lagrangian formulations only. In all previous cases we had $N+1$ such formulations for N-component systems.

5. Conclusions

We have studied here restricted flows (2.12) of cKdV and cHD hierarchies which come from the energy-dependent Schrödinger spectral problem (2.1). There is a multitude of restricted flows which depend on the choice of vector field K_r and Hamiltonian structure B_m in (2.12*b*). We have determined here these restricted flows for which (2.12*b*) is explicitly solvable for u in terms of ϕ. Restricted flows of the first two vector fields K_0, K_1 have been determined in complete generality (for arbitrary number of fields u_r) and have been identified with known potential Newton equations. These potentials are the Jacobi and Neumann families of potentials separable in generalized elliptic and spherical-conical systems of coordinates correspondingly. Our results establish a close link between restricted (stationary) flows of the energy-dependent Schrödinger spectral problem and the classical separability theory of the Hamilton–Jacobi equation.

Due to the multi-Hamiltonian nature of cKdV and cHD hierarchies there are many equivalent forms of the restrictions (2.12*b*). These provide several Lagrangian formulations for the restricted flows and in a quite natural way lead to finite-dimensional bi-Hamiltonian systems. This is of great interest since the bi-Hamiltonian formulation for finite-dimensional integrable systems has not yet been developed. These questions have been discussed in [1, 2]. The results of this paper provide a lot of new material for such studies. We only discuss here (in appendix 2) a new Lagrangian and Hamiltonian formulation of the Neumann system which is equivalent (in an extended phase space) to Moser's formulation [3].

Recently Zeng and Li published a paper [8] where restricted flows of the energy-dependent Schrödinger spectral problem have been studied by the use of a somewhat different method [9]. Their results correspond to part of the results of section 3.2 when $\varepsilon = 1$, $r = 1$.

Appendix 1. The Neumann family of symmetric potentials

The spherical-conical coordinates $\zeta_1, \ldots, \zeta_{n-1}, q$ are defined by the rational equation

$$\sum_{k=1}^{n} \frac{q_k^2}{z-\alpha_k} = q^2 \left\{ \prod_{j=1}^{n-1} (z-\zeta_j) \Big/ \prod_{k=1}^{n} (z-\alpha_k) \right\} \qquad q^2 = q_1^2 + \ldots + q_n^2$$

where $0 < \alpha_1 < \ldots < \alpha_n$ and $q_1, \ldots, q_n$ are the Cartesian coordinates. The general form of a potential separable in spherical-conical coordinates $\zeta_1, \ldots, \zeta_{n-1}, q$ is given by the Stäckel theorem [10] but for physical applications more important is the form of separable potentials expressed in terms of Cartesian coordinates $q_1, \ldots, q_n$. A general separable potential cannot be expressed explicitly through the Cartesian coordinates because high-order algebraic equations have to be solved. A special class of permutationally symmetric potentials separable in spherical-conical coordinates, for which explicit Cartesian expressions exist, has been found in [7]. These potentials are called Neumann potentials because they remain separable when constrained to a sphere and one of them is the Neumann harmonic potential.

The Cartesian form of the complete set of commuting integrals of motion is the key to the characterization of the Neumann potentials. It has been proved in [7, 11] that a natural Hamiltonian is separable in the spherical-conical coordinates if and only if it has $n-1$ global functionally independent and commuting integrals of motion of the form

$$K_i(q,p)=\sum_{j=1,j\neq i}^{n}\frac{l_{ij}^2}{\alpha_i-\alpha_j}+k_i(q)=P_i(q,p)+k_i(q) \qquad i=1,\ldots,n-1 \tag{A1}$$

where $l_{ij}=q_ip_j-q_jp_i$, $i,j=1,\ldots,n$.

These integrals are quadratic in momenta, and $k_i(q)$ are such functions of $q=(q_1,\ldots,q_n)$ which guarantee commutativity of K_i. Neumann potentials are derived from the following proposition.

Proposition 1. Let

$$K_i^{(m)}(q,p)=P_i(q,p)+k_i^{(m)}(q) \qquad i=1,\ldots,n \tag{A2}$$

$m=1,2,\ldots$, be a sequence of n-tuples of functions defined recursively by

$$k_i^{(m+1)}=-q_i^2\left(\sum_{j=1}^{n}\alpha_jk_j^{(m)}\right)+q^2\alpha_ik_i^{(m)}(q) \qquad k_i^{(1)}=q_i^2. \tag{A3a}$$

Then $K_i^{(m)}$ commute for all $m=1,2,\ldots$. *Moreover the recursion relation* (A3a) *can* be inverted

$$k_i^{(m)}=-q_i^2(\alpha_iq^2N_1)^{-1}\left(\sum_{j=1}^{n}\alpha_j^{-1}k_j^{(m+1)}\right)+q^{-2}\alpha_i^{-1}k_i^{(m+1)}(q) \qquad \text{for } m=-1,2,\ldots$$

$$k_i^{(0)}=q_i^2(\alpha_iN_1)^{-1} \qquad N_1=(q\mathscr{A}^{-1}q) \qquad \mathscr{A}=\mathrm{diag}(\alpha_1,\ldots,\alpha_n) \tag{A3b}$$

and integrals $K_i^{(m)}$ commute for $m=0,-1,-2,\ldots$.

Only $n-1$ integrals of the type (A2) are independent since $K_1^{(m)}+\ldots+K_n^{(m)}=0$. However, functions $k_i^{(m)}$ are rational homogeneous functions of degree $2m$ and new functions $\bar{k}_i^{(m)}=q^{-2m}k_i^{(m)}$ are of degree 0. New integrals $\bar{K}_i^{(m)}=P_i(q,p)+\bar{k}_i^{(m)}$ also commute and moreover commute with $(qp)=q_1p_1+\ldots+q_np_n$. The Hamiltonian

$$H(q,p)=\tfrac{1}{2}q^{-2}\left[\sum_{i=1}^{n}\alpha_i\bar{K}_i^{(m)}+(qp)^2\right]=\tfrac{1}{2}(pp)+\tfrac{1}{2}q^{-2}\sum_{i=1}^{n}\alpha_i\bar{k}_i^{(m)}$$

has the natural kinetic energy term and homogeneous of degree -2 potential

$$V^{(m)}=\tfrac{1}{2}q^{-2}\sum_{i=1}^{n}\alpha_i\bar{k}_i^{(m)}. \tag{A4}$$

Note that if we add to $V^{(m)}$ any spherically symmetric function $f(q^2)$ then the new potential has the same integrals $\bar{K}_i^{(m)}$ and remains completely integrable. Recursion relations (A3a, b) define through (A4) two separate families of integrable potentials. Potentials of the upward family read:

$$\tfrac{1}{2}q^{-4}(q\mathscr{A}q) \qquad -\tfrac{1}{2}q^{-4}(q\mathscr{A}^2q)-\tfrac{1}{2}q^{-6}(q\mathscr{A}q)^2$$

$$\tfrac{1}{2}q^{-4}(q\mathscr{A}^3q)-\tfrac{1}{2}q^{-6}(q\mathscr{A}^2q)(q\mathscr{A}q)+\tfrac{1}{2}q^{-8}(q\mathscr{A}q)^3\ldots.$$

The first potential, when constrained to a sphere $q^2=\text{constant}$, gives the well known

Neumann system. The first few potentials of the downward family are:

$$\frac{1}{(q\mathscr{A}^{-1}q)} \qquad q^{-2}-\frac{(q\mathscr{A}^{-2}q)}{(q\mathscr{A}^{-1}q)^2} \qquad -\frac{(q\mathscr{A}^{-3}q)}{(q\mathscr{A}^{-1}q)^2}+\frac{(q\mathscr{A}^{-2}q)^2}{(q\mathscr{A}^{-1}q)^3}$$

$$\frac{(q\mathscr{A}^{-4}q)}{(q\mathscr{A}^{-1}q)^2}-\frac{2(q\mathscr{A}^{-2}q)(q\mathscr{A}^{-3}q)}{(q\mathscr{A}^{-1}q)^3}+\frac{(q\mathscr{A}^{-2}q)^3}{(q\mathscr{A}^{-1}q)^4}, \ldots$$

The spherically symmetric term q^{-2} in the second potential is not essential since all integrable potentials of the Neumann family are determined modulo the additive term $f(q^2)$.

Appendix 2. Lagrangian formulation and Hamiltonian structure of the Neumann system

Lagrangian formulation

The restriction $0=B_1(\phi\Lambda^{-1}\phi)$ leads to the second form (3.3) of the Neumann system. It is generated by the Lagrangian (3.4). For the proper identification of the forms (3.1) and (3.3) we have to extend the phase space of variables ϕ, ϕ_x by the integration constants c_0 and d_0 correspondingly. Then the standard form of the Neumann system reads

$$\begin{aligned} &0=\phi_{xx}+\left[\frac{(\phi_x\phi_x)}{(\phi\phi)}+\frac{(\phi\Lambda\phi)}{(\phi\phi)}-\Lambda\right]\phi \qquad c_{0x}=0 \\ &(\phi\phi)=c_0. \end{aligned} \tag{A5}$$

For the second formulation we note first that by multiplying (3.3) by $\phi_x\Lambda^{-1}$, $\phi\Lambda^{-1}$ and ϕ we get

$$0=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(\phi_x\Lambda^{-1}\phi_x)-\frac{1}{2}\frac{(\phi\Lambda^{-1}\phi_x)^2+d_0}{(\phi\Lambda^{-1}\phi)^{-1}}-\frac{1}{2}(\phi\phi)\right]=\frac{\mathrm{d}}{\mathrm{d}x}\bar{E}(\phi,\phi_x,d_0) \tag{A6a}$$

$$0=-2[\bar{E}(\phi,\phi_x,d_0)+(\phi\phi)] \qquad \text{hence } (\phi\phi) \text{ is also an integral} \tag{A6b}$$

$$\frac{(\phi\Lambda^{-1}\phi_{xx})}{(\phi\Lambda^{-1}\phi)}=\frac{(\phi\phi_{xx})}{(\phi\phi)}+\frac{(\phi\phi)}{(\phi\Lambda^{-1}\phi)}-\frac{(\phi\Lambda\phi)}{(\phi\phi)}. \tag{A6c}$$

The expression $\bar{E}$ is the energy integral of (3.3). Equation (A6*b*) is a first-order constraint satisfied by all solutions of (3.3) and it implies that $(\phi\phi)$ is also an integral of motion of (3.3). Equation (A6*c*) is satisfied identically by solutions of (3.3) and (3.1). We shall write the system (3.3) as

$$\begin{aligned} &0=\phi_{xx}-\left[\frac{(\phi\Lambda^{-1}\phi_{xx})+(\phi_x\Lambda^{-1}\phi_x)}{(\phi\Lambda^{-1}\phi)^{-1}}-\frac{(\phi\Lambda^{-1}\phi_x)^2+d_0}{(\phi\Lambda^{-1}\phi)^{-2}}+\Lambda\right]\phi \qquad d_{0x}=0 \\ &\bar{E}(\phi,\phi_x,d_0)+(\phi\phi)=0. \end{aligned} \tag{A7}$$

To see the equivalence of (A7) with (A5) we rewrite (A7) in the form

$$\begin{aligned} 0&=\phi_{xx}-\frac{(\phi\Lambda^{-1}\phi_{xx})}{(\phi\Lambda^{-1}\phi)}\phi-\frac{2\bar{E}}{(\phi\Lambda^{-1}\phi)}\phi-\frac{(\phi\phi)}{(\phi\Lambda^{-1}\phi)}-\Lambda\phi \\ &=\phi_{xx}-\frac{(\phi\phi_{xx})}{(\phi\phi)}\phi-\frac{2(\bar{E}+(\phi\phi))}{(\phi\Lambda^{-1}\phi)}\phi+\frac{(\phi\Lambda\phi)}{(\phi\phi)}\phi-\Lambda\phi \\ &=\phi_{xx}+\left[\frac{(\phi_x\phi_x)}{(\phi\phi)}+\frac{(\phi\Lambda\phi)}{(\phi\phi)}-\Lambda\right]\phi \end{aligned}$$

where the last equality follows from the equation of constraints $\bar{E}+(\phi\phi)=0$ and its second derivative $(\phi\phi_{xx})+(\phi_x\phi_x)=0$. Thus every solution of the system (A7) with a fixed value of the constant d_0 also fulfils the Neumann system (A5) with $c_0=(\phi\phi)=-\bar{E}(\phi,\phi_x,d_0)$ since $\bar{E}$ is an integral of motion for (A7). Conversely, the Lagrangian system (A7) has a unique solution of the initial value problem $\phi(0)$, $\phi_x(0)$, and for any trajectory of the Neumann system (A5) staying on $(\phi\phi)=c_0$ we can determine the value of d_0 (from $\bar{E}(\phi,\phi_x,d_0)=-c_0$) so that the corresponding solution of (A7) with fixed $\phi(0)$, $\phi_x(0)$ coincides with the starting solution of the Neumann system (A5).

Hamiltonian structure

For a constrained system, like the Neumann system, there may exist many Hamiltonian formulations which lead to the same equations on the manifold of constraints. Among these the most interesting are those which can be used to prove complete integrability of the constrained system.

A natural source for guessing an appropriate Hamiltonian is the separability of the Neumann system in spherical-conical coordinates and the corresponding set of commuting integrals of motion

$$K_i=\sum_{j=1,j\neq i}^{n}\frac{l_{ij}^2}{\lambda_i-\lambda_j}-q_i^2 \qquad l_{ij}=q_ip_j-q_jp_i \qquad i,j=1,\ldots,n.$$

These integrals have been used by Moser [3] to construct the integrable Hamiltonian

$$H_{\rm M}=\tfrac{1}{2}[(pp)(qq)-(qp)^2-(q\Lambda q)]=\tfrac{1}{2}\sum_{i=1}^{n}\lambda_iK_i$$

where $q=(q_1,\ldots,q_n)$, $p=(p_1,\ldots,p_n)$. It yields the Neumann system on the surface of constraints $F_1=(qq)-1=0$, $F_2=(qq_x)=(qp)=0$ and remains integrable on this surface.

In order to see how Moser's Hamiltonian description of the Neumann system follows from the Lagrangian (3.4) we have to use the parameter-dependent Hamiltonian

$$H_{\rm M}=\frac{1}{2c_0}[(pp)(qq)-(qp)^2-(q\Lambda q)]$$

constrained to $F_1=(qq)-c_0=0$, $F_2=(qp)=0$. Then the requirement of invariance of the constraints F_1, F_2 under the modified Hamiltonian $H^*=H_{\rm M}+\mu_1F_1+\mu_2F_2$:

$$0=\dot{F}_1=\{F_1,H^*\}$$

$$0=\dot{F}_2=\{F_2,H^*\} \qquad \text{on } F_1=0=F_2$$

leads to the Hamiltonian

$$H^*=H_{\rm M}+\frac{(q\Lambda q)}{2(qq)}F_1=\frac{1}{2c_0}[(pp)(qq)-(qp)^2]-\frac{c_0(q\Lambda q)}{2(qq)}$$

which (accidentally) commutes with F_1, F_2 everywhere (not only on $F_1=0=F_2$). The Hamilton equations of H^* on the manifold $F_1=0=F_2$ are

$$q_x = p \qquad p_x = \Lambda q - \frac{1}{c_0}[(pp)+(q\Lambda q)]q. \tag{A8}$$

They give

$$0 = q_{xx} + \frac{1}{c_0}\left[\frac{(qq)(pp)}{c_0} + \frac{c_0(q\Lambda q)}{(qq)} - \frac{(pq)^2}{c}\right] q - \Lambda q$$

which, after using $F_1=0=F_2$, coincides with the Neumann system on the sphere $(qq)=c_0$.

A second Hamiltonian formulation for the Neumann system follows from the Lagrangian (3.4) which generates the Legendre transformation

$$\bar{q} = \phi \qquad \bar{p} = \frac{\partial \mathscr{L}}{\partial \phi_x} = \Lambda^{-1}\phi_x - \frac{(\phi\Lambda^{-1}\phi_x)}{(\phi\Lambda^{-1}\phi)}\Lambda^{-1}\phi.$$

The corresponding Hamiltonian reads

$$\bar{H} = (\bar{p}\phi_x) - \mathscr{L} = \tfrac{1}{2}(\bar{p}\Lambda\bar{p}) - \tfrac{1}{2}(\bar{q}\bar{q}) - \frac{2d_0}{(\bar{q}\Lambda^{-1}\bar{q})}.$$

This Legendre transformation is degenerate (compare (3.23)): $G_1 = (\bar{p}\bar{q}) = 0$ which means that we have a constraint system with the primary constraint $G_1=0$. In order to determine the complete manifold of constraints and the modified Hamiltonian $\bar{H}^* = \bar{H} + \mu G_1$ we apply the Dirac theory [12]. The equation

$$\{G_1, \bar{H}\} = \{G_1, \bar{H}^*\} = 2\bar{H} + 2(\bar{q}\bar{q}) = 2G_2$$

defines G_2 and subsequently

$$0 = \{G_2, \bar{H}^*\} = 2(\bar{p}\Lambda\bar{q}) + \mu(-2\bar{H})$$

determines μ. Hence

$$\bar{H}^* = \bar{H} + \frac{(\bar{p}\Lambda\bar{q})}{\bar{H}} G_1 = \tfrac{1}{2}(\bar{p}\Lambda\bar{p}) - \tfrac{1}{2}(\bar{q}\bar{q}) - \frac{2d_0}{(\bar{q}\Lambda^{-1}\bar{q})} + \frac{(\bar{p}\bar{q})(\bar{p}\Lambda\bar{q})}{\bar{H}}$$

since $\bar{H} = -(\bar{q}\bar{q})$ on the constraints. Thus on the constraints $G_1 = 0 = G_2$ we get equations

$$\bar{q}_x = \Lambda\bar{p} - \frac{(\bar{p}\Lambda\bar{q})}{(\bar{q}\bar{q})}\bar{q}$$

$$\bar{p}_x = \bar{q} - \frac{4d_0}{(\bar{q}\Lambda^{-1}\bar{q})^2}\Lambda^{-1}\bar{q} - \frac{(\bar{p}\Lambda\bar{q})}{\bar{H}}\bar{p}.$$

These equations imply (A8) if

$$q = \bar{q} \qquad p = \Lambda\bar{p} - \frac{(\bar{p}\Lambda\bar{q})}{(\bar{q}\bar{q})}\bar{q} \qquad c_0 = -\bar{H}(\bar{q}, \bar{p}, d_0)$$

since from $G_2 = 0$ we have $\bar{H} = -(\bar{q}\bar{q})$.

References

[1] Antonowicz M and Rauch-Wojciechowski S 1990 *Phys. Lett.* **147A** 455

[2] Antonowicz M and Rauch-Wojciechowski S 1990 How to construct finite-dimensional bi-Hamiltonian systems from soliton equations: Jacobi integrable potentials *Preprint* LiTH-MAT-R-90-35

[3] Moser J 1978 Various aspects of integrable Hamiltonian systems *Proc. CIME Conf. (Bressanone, Italy, 1978)*

[4] Antonowicz M and Fordy A P 1989 *Commun. Math. Phys.* **124** 465

[5] McKean H and Trubowitz E 1976 *Comm. Pure Appl. Math.* **29** 143

[6] Choodnovsky D V and Choodnovsky G V 1978 *Lett. Nuovo Cimento* **22** 47

[7] Wojciechowski S 1986 *Proc. of SMS: Systèmes Dynamiques Nonlinéaires* ed P Winternitz (Montreal: Presse Université Montreal) p 294

[8] Zeng Y and Li Y 1990 *J. Math. Phys.* **31** 2835

[9] Zeng Y and Li Y 1989 *J. Math. Phys.* **30** 1679

[10] Stäckel P 1891 Uber die Integration der Hamilton–Jacobishen Differentialgleichung mittelst Separation der Variablen. Habilitationsschrift, Halle

[11] Marshall I and Wojciechowski S 1988 *J. Math. Phys.* **29** 1338

[12] Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Yeshiva University Press)